

\mathcal{F} -FUNCTIONAL AND GEODESIC STABILITY

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ABSTRACT. We consider canonical metrics on Fano manifolds. First we introduce a norm-type functional on Fano manifolds, which has Kahler-Einstein or Kahler-Ricci soliton as its critical point and the Kahler-Ricci flow can be viewed as its (reduced) gradient flow. We then obtain a natural lower bound of this functional. As an application, we prove that Kahler-Ricci soliton, if exists, maximizes Perelman's μ -functional without extra assumptions. Second we consider a conjecture proposed by S.K. Donaldson in terms of \mathcal{K} -energy. Our simple observation is that \mathcal{F} -functional, as \mathcal{K} -energy, also integrates Futaki invariant. We then restate geodesic stability conjecture on Fano manifolds in terms of \mathcal{F} -functional. Similar pictures can also be extended to Kahler-Ricci soliton and modified \mathcal{F} -functional.

1. INTRODUCTION

Let (M, L) be a polarized projective manifold where L is a positive line bundle over M . The Yau-Tian-Donaldson conjecture [33, 29, 17] in Kahler geometry asserts that the Kahler class $c_1(L)$ contains a Kahler metric of constant scalar curvature metric if and only if (M, L) is K-stable. In this paper we consider Fano manifolds, hence L is the anti-canonical bundle, denoted as $-K_M$. In this case, a constant scalar curvature in the class $c_1(-K_M)$ has to be a Kähler-Einstein metric. Kahler metrics with constant scalar curvature are special cases of Calabi's extremal metric [8]. In 1980s, Calabi introduced a beautiful variational approach to find a canonical representative for a polarization (M, L) , or even more general for a fixed Kahler class $(M, [\omega])$. He considered a natural energy functional of metrics, the L^2 norm of full curvature tensor, which is equivalent to the Calabi functional (L^2 norm of scalar curvature) and an extremal metric is a critical point of this functional when restricted in $(M, [\omega])$. Futaki [19] introduced the Futaki invariant (see Calabi [9], Bando [1] also) and an extremal metric is a Kahler metric with constant scalar curvature precisely when the Futaki invariant of $(M, [\omega])$ vanishes. The Yau-Tian-Donaldson conjecture were generalized to extremal metrics naturally by T. Mabuchi [24] and G. Székelyhidi [28].

When $(M, -K_M)$ is Fano, there is another natural canonical metric called Kahler-Ricci soliton. Ricci solitons were defined by R. Hamilton in the study of Ricci flow [20] and it plays a significant role in the theory of Ricci flow. A Kahler-Ricci soliton on a Fano manifold is Kahler-Einstein precisely when the Futaki invariant vanishes and hence Kahler-Ricci soliton is also a natural extension of Kahler-Einstein metrics. However when the Futaki invariant is nonzero, a Kahler-Ricci soliton is different from an extremal metric on $(M, -K_M)$. It is then natural to extend Yau-Tian-Donaldson conjecture to

cover Kahler-Ricci solitons. This is one of main motivations of the present paper.

In his study of Hamilton's Ricci flow, Perelman [25] introduced many revolutionary ideas, including the well-known entropy functionals, which lead him to the solution of the Poincare conjecture and Thurston's geometrization conjecture. Ricci flow is the gradient flow of Perelman's μ -functional (modulo diffeomorphisms). On Fano manifolds, Kahler-Ricci solitons are critical points of Perelman's μ -functional. A Kahler-Einstein metric, if exists, maximizes μ -functional. Kahler-Ricci soliton is also expected to maximize μ -functional and it is actually the case [32], for example, if we consider only metrics which are invariant under the action of maximal torus of automorphism group (invariant metrics for short). Note that the invariant assumption can actually be made more specific for imaginary part of extremal vector field studied in Tian-Zhu [31].

Our first observation is to consider a functional (this quantity has been studied along the Kahler-Ricci flow in literature) on Fano manifolds. We show that Kahler-Ricci flow is a (reduced) gradient flow of this functional and in particular, a Kahler-Ricci soliton is a critical point of this functional. We then consider a lower bound of this functional and this turns out to be rather straightforward for invariant metrics. Without invariant assumption, we need to consider a natural geometric structure of the space of Kahler metrics, studied by Mabuchi [22], Semmes [27] and Donaldson [16]. The key notion is the geodesic segments and the geodesic rays, in the manner of X.-X. Chen [11, 12]. We then obtain a natural lower bound, called H-invariant which was studied in [32] and this result relies on the key technique fact of the convexity of Ding's \mathcal{F} -functional [15], established by Berndtsson [5, 6]. As a direct application, we prove that Kahler-Ricci soliton, if exists, maximizes Perelman's μ -functional, without invariant assumption.

Donaldson [16] formulated a conjecture relating existence of constant scalar curvature with the geometric structure of the space of Kahler potentials, in particular with the limit behavior of (derivative of) Mabuchi's \mathcal{K} -energy (defined in [22]) along geodesic rays. This conjecture, on one hand, is a natural extension of Hilbert-Mumford criterion in Geometric Invariant Theory to infinite dimensional case. In particular, Donaldson's conjecture is expected to be equivalent to K -stability conjecture in terms of the Donaldson-Futaki invariant of test configuration formulated by Donaldson [17], which is of algebro-geometric nature. On the other hand, it is very intuitive to understand Donaldson's conjecture in terms of critical points of the (formally) convex functional—Mabuchi's \mathcal{K} -energy. The constant scalar curvature metric is a critical point of \mathcal{K} -energy (actually minimizer) and \mathcal{K} -energy is (formally) convex in terms of geodesics in the space of Kahler potentials. On Fano manifolds, Tian [29] has proved that the existence of Kahler-Einstein metric is equivalent to the properness of \mathcal{K} -energy (and also \mathcal{F} -functional) and he proved further that the existence of Kahler-Einstein metric implies K -stability for special degenerations. Tian's properness condition is formulated as a Morse-Trudinger type inequality; recently Chen [13]

considered the properness of \mathcal{K} energy in terms of geodesic distance. Nevertheless it is natural to expect that the geodesic stability formulated in terms of \mathcal{K} -energy is equivalent to the properness of K-energy in terms of geodesic distance (we abuse the notation of properness here). This can actually be understood intuitively as follows: if the space of Kahler potentials were finite dimensional, then properness of \mathcal{K} -energy would be equivalent to properness of \mathcal{K} -energy in each direction since a unit ball (in finite dimensional space) is compact; while the properness in each direction is exactly what geodesic stability means. Donaldson's conjecture naturally leads to the geodesic stability which was first introduced by Chen [12] in terms of ρ -invariant along geodesic rays (see Phong-Sturm [26] also for some related work). Chen [13] made one further step to study the lower bound of \mathcal{K} -energy and partially confirmed Donaldson's conjecture.

Our second observation is that on Fano manifold, \mathcal{F} -functional also integrates the Futaki invariant as \mathcal{K} -energy in the sense of [23]. It is then natural to consider the geodesic stability in terms of \mathcal{F} -functional. The key advantage of \mathcal{F} -functional is that it is actually convex for geodesic rays with very weak regularity. This removes the main technical obstacle caused by the rather weak regularity of geodesic segments and geodesic rays. This observation can then be extended to cover Kahler-Ricci soliton using modified \mathcal{F} -functional and modified Futaki invariant, studied by Tian-Zhu [30, 31]. Chen's series work on geodesic stability [12, 13] has definite influence on the present work; nevertheless, it seems to be interesting to use \mathcal{F} -functional instead on Fano manifolds.

This key advantage of \mathcal{F} -functional has already been explored by Berndtsson [6] (see Berman [3] also). Very recently Berman [4] proved that the existence of Kahler-Einstein metric implies K-polystability (his results hold for more general Q -Fano variety), using the convexity of \mathcal{F} -functional along weak geodesic rays. One of his key ideas is that the numerical invariant (Donaldson-Futaki invariant) used to test stability for a test configuration can be computed in terms of asymptotes of Ding's \mathcal{F} -functional along a weak geodesic ray determined by the test configuration. Berman's result shows that the geodesic stability in terms of geodesic rays is essentially equivalent to the K-stability.

We organize the paper as follows. In Section 2 we study the \mathcal{H} -functional and prove that it has a natural lower bound. In Section 3 we discuss the geodesic stability in terms of \mathcal{F} -functional on Fano manifolds. In Section 4 we then extend the discussion of geodesic stability to Kahler-Ricci soliton.

Acknowledgement: The author is very grateful to Song Sun for numerous enlightening discussions and suggestions. The author also thanks Prof. X.-X. Chen for constant encouragements.

2. KÄHLER-RICCI SOLITON, \mathcal{H} FUNCTIONAL AND \mathcal{F} FUNCTIONAL

Let (M, ω_0) be a compact Fano manifold. For any Kähler metric $\omega \in [\omega_0]$, the Ricci potential h of ω is defined to be

$$Ric(\omega) - \omega = \sqrt{-1}\partial\bar{\partial}h,$$

with the normalization condition

$$\int_M e^h \omega^n = \int_M \omega^n = V.$$

We define a \mathcal{H} functional as follows,

$$(2.1) \quad \mathcal{H}(\omega) = \int_M h e^h \omega^n.$$

By Jensen's inequality (apply to the convex function $\varphi(x) = x \log x, x > 0$),

$$V^{-1} \mathcal{H}(\omega) \geq \log \left(\frac{1}{V} \int_M e^h \omega^n \right) \frac{1}{V} \int_M e^h \omega^n = 0.$$

Hence $\mathcal{H}(\omega)$ is a norm-like functional and it is zero precisely when $h = 0$, namely ω is a Kahler-Einstein metric.

We compute its first variation as follows. Note that we can assume

$$\omega = \omega_\phi = \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi.$$

Suppose the variation of ϕ is given by $\delta \phi$, and then we can compute,

$$\delta h = -\Delta \delta \phi - \delta \phi.$$

Note that we also have the normalization condition that

$$\int_M \delta \phi e^h \omega^n = 0.$$

We compute

$$(2.2) \quad \begin{aligned} \delta \mathcal{H}(\omega) &= \int_M \left(-(\Delta \delta \phi + \delta \phi) e^h - h e^h (\Delta \delta \phi + \delta \phi) + h e^h \Delta \delta \phi \right) \omega^n \\ &= - \int_M \delta \phi (\Delta h + |\nabla h|^2 + h) e^h \omega^n. \end{aligned}$$

The Euler-Lagrangian equation is given by

$$(2.3) \quad \Delta h + |\nabla h|^2 + h = \text{constant}.$$

Clearly the constant above is given exactly by \mathcal{H} . Note that (2.3) implies that ∇h is a real holomorphic vector field and hence the critical point of $\mathcal{H}(\omega)$ is a Kähler-Ricci soliton. In particular, along Kahler-Ricci flow

$$\frac{\partial \omega}{\partial t} = \omega - \text{Ric}(\omega),$$

One can compute directly that

$$(2.4) \quad \frac{\partial \mathcal{H}}{\partial t} = - \int_M (|\nabla h|^2 - (h - \mathcal{H})^2) e^h \omega^n \leq 0.$$

The equality holds exactly when ω is a Kähler-Ricci soliton.

Remark 2.1. By (2.2) the gradient flow of $\mathcal{H}(\omega)$ is given by

$$\frac{\partial \phi}{\partial t} = \Delta h + |\nabla h|^2 + h$$

and it is a fourth-order equation. However, since the operator

$$L_h u = -(\Delta u + \nabla u \nabla h + u)$$

is a self-adjoint positive operator with respect to $e^h \omega^n$, we can hence reduce the gradient flow by second order by studying

$$\frac{\partial \phi}{\partial t} = -h,$$

which is exactly the Kähler-Ricci flow on potential level. In this sense, the Kähler-Ricci flow on Fano manifolds can be viewed as a (reduced) gradient flow of $\mathcal{H}(\omega)$. The formal picture resembles that of the Calabi flow for the Calabi energy and extremal metric. In this sense, for Kähler-Ricci solitons, $\mathcal{H}(\omega)$ plays the role of the Calabi energy for extremal metrics.

If the automorphism group of M contains a connected (maximal) compact subgroup G , then there is another natural lower bound on \mathcal{H} functional. Such a lower bound is an invariant of $(M, [\omega_0])$ studied in [32], which itself is related to the modified Futaki invariant defined in [31]. Suppose ω is a G -invariant metric. Then there is a Lie algebra homomorphism from $Lie(G)$ to the functions on M , under Poisson bracket. Let $\xi \in Lie(G)$ and let θ_ξ be the corresponding Hamiltonian. We also assume a normalization condition for θ_ξ ,

$$\int_M e^{\theta_\xi} \omega^n = V.$$

Define the integral

$$(2.5) \quad H(\xi) = \int_M \theta_\xi e^h \omega^n.$$

It was proved [32] that H is independent of G -invariant metrics in $[\omega_0]$ and H is a concave function in $Lie(G)$ with a unique maximizer $\xi_0 \in Lie(G)$ and ξ_0 is the imaginary part of the extremal vector field X studied in [31]. Then we have,

Proposition 2.2. *Let ω be a G -invariant metric, then*

$$\mathcal{H}(\omega) \geq \sup_{\xi \in Lie(G)} H(\xi) = H(\xi_0).$$

Proof. We only need to show

$$\int_M (h - \theta_\xi) e^h \omega^n \geq 0.$$

This is actually an elementary property of convex functions and we have

$$(2.6) \quad \int_M (h - f) e^h \omega^n \geq 0$$

provided that the function f satisfies the normalization condition

$$\int e^f \omega^n = V.$$

Suppose $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a (smooth) convex function, then for any $f, g : M \rightarrow \mathbb{R}$, we define $F : [0, 1] \rightarrow \mathbb{R}$ by

$$F(t) = \int_M \varphi(tf + (1-t)g) \omega^n - t \int_M \varphi(f) \omega^n - (1-t) \int_M \varphi(g) \omega^n.$$

Clearly $F(0) = F(1) = 0$ and

$$F''(t) = \int_M \varphi''(tf + (1-t)g)(f-g)^2 \omega^n \geq 0.$$

It follows that $F(t) \leq 0$ and $F'(0) \leq 0, F'(1) \geq 0$. Now taking $\varphi(x) = e^x, f = \theta_\xi, g = h$, we get

$$F'(0) = \int_M e^h(\theta_\xi - h)\omega^n \leq 0.$$

□

It would certainly be interesting to understand the lower bound of $\mathcal{H}(\omega)$ for all the metrics in $[\omega_0]$, not only for invariant metrics (this is related to geodesic stability of Kahler-Ricci solitons). For the Calabi energy, such a lower bound was proved only recently (for non-invariant metrics) by X.-X. Chen [12] using deep estimates of homogeneous complex Monge-Ampere equations in the space of Kahler potentials [11, 14] and by S. Donaldson [18] using finite dimensional approximations. We shall postpone the discussion related to geodesic stability and give a direct proof for the lower bound of $\mathcal{H}(\omega)$ for non-invariant metrics.

The method mimics Chen's approach [12] for the Calabi functional, which essentially relied on the convexity of Mabuchi's K-energy along geodesic. However, the derivative of K-energy is not even well-defined for $C^{1,1}$ geodesic and so further regularity results are needed and we refer to Chen's paper [12] for delicate details. In our case, however, the convexity of \mathcal{F} functional is actually well-defined and H -invariant is closely related to the derivative of \mathcal{F} -functional. Hence we can obtain,

Theorem 2.3. *For any metric $\omega \in [\omega_0]$, then*

$$(2.7) \quad \mathcal{H}(\omega) \geq \sup_{\xi \in \text{Lie}(G)} H(\xi) = H(\xi_0).$$

Before we proceed to the proof, we explain a little bit the relation of Ding's \mathcal{F} -functional [15] and H -invariant. Recall \mathcal{F} functional on Fano manifolds can be defined through its derivative as

$$\frac{d\mathcal{F}}{dt} = - \int_M \dot{\phi} \omega_\phi^n + \int_M \dot{\phi} e^{h_\phi} \omega_\phi^n.$$

A direct computation implies that \mathcal{F} functional is convex (along smooth geodesics of Kahler potentials),

$$\frac{d^2}{dt^2} \mathcal{F} = \int_M (\ddot{\phi} - |\nabla \dot{\phi}|^2)(e^h - 1)\omega^n + \int_M (|\nabla \dot{\phi}|^2 - \dot{\phi}^2)e^h \omega^n \geq 0,$$

where the second term is nonnegative by the modified Poincare inequality on Fano manifolds and we have assumed the normalization conditions

$$\int_M e^h \omega^n = V, \int_M \dot{\phi} e^h \omega^n = 0.$$

It is not hard to see that really the piece $\int_M \dot{\phi} e^h \omega^n$ provides the convexity (see (2.10)) and the other piece $-\int \dot{\phi} \omega^n$ can be considered as a normalization

condition. For \mathcal{F} -functional, we can then associate an invariant for $\xi \in \text{Lie}(G)$,

$$(2.8) \quad F(\xi) = \int_M \theta_\xi e^h \omega^n - \int_M \theta_\xi \omega^n$$

and \mathcal{F} -functional integrates this invariant in the sense that, if σ_t^* is the holomorphic transformations generated by $X = J\xi + \sqrt{-1}\xi$, then

$$\frac{d\mathcal{F}(\sigma_t^* \omega)}{dt} = F(\xi).$$

Clearly by Jensen's inequality,

$$F(\xi) - H(\xi) = V \log \left(V^{-1} \int_M e^{\theta_\xi} \omega^n \right) - \int_M \theta_\xi \omega^n \geq 0.$$

Proposition 2.4. *$F(\xi)$ is just the well-known Futaki invariant. In particular, \mathcal{F} -functional integrates Futaki-invariant.*

Proof. Note that θ satisfies the following equation,

$$\Delta \theta_\xi + \nabla \theta_\xi \cdot \nabla h + \theta_\xi = a,$$

where the constant a is given by

$$\int_M \theta_\xi e^h \omega^n = aV.$$

Recall the Futaki invariant [19] is defined to be

$$F_0(\xi) = \int_M \nabla \theta_\xi \cdot \nabla h \omega^n.$$

We can then compute

$$\int_M (\Delta \theta_\xi + \nabla \theta_\xi \cdot \nabla h + \theta_\xi) \omega^n = F_0(\xi) + \int_M \theta_\xi \omega^n = aV.$$

It then follows that

$$F_0(\xi) = aV - \int_M \theta_\xi \omega^n = \int_M \theta_\xi e^h \omega^n - \int_M \theta_\xi \omega^n = F(\xi).$$

□

Remark 2.5. Note that, for any (smooth) function $f : \mathbb{R} \rightarrow \mathbb{R}$ we can choose a normalization condition (for geodesics)

$$\int_M f(\dot{\phi}) \omega^n = 0,$$

and such a normalization condition is preserved along geodesics. Similarly, we can define a family of invariants $H_f(\xi)$ as

$$H_f(\xi) = \int_M \theta_\xi e^h \omega^n$$

with respect to a normalization condition, for a given function $f : \mathbb{R} \rightarrow \mathbb{R}$,

$$\int_M f(\theta_\xi) \omega^n = 0.$$

However, only the choice of $f(\dot{\phi}) = \dot{\phi}$ (or f is a linear function) can be integrated out to define a functional (\mathcal{F} -functional) on the space of Kahler potentials.

Now we are ready to prove Theorem 2.3.

Proof. For any smooth Kähler metric $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi \in [\omega_0]$, the Ricci potentials are related by

$$h = h_0 - \log \frac{\omega^n}{\omega_0^n} - \phi + \text{constant}.$$

We always choose the normalization condition for Ricci potential such that

$$\int_M e^h \omega^n = V.$$

In particular we get the relation between two measure as follows,

$$e^h \omega^n = \lambda e^{h_0 - \phi} \omega_0^n$$

for some positive constant λ . The advantage of the above relation is that the measure $e^h \omega^n$ is well defined for any closed positive $(1, 1)$ current $\omega \in [\omega_0]$ provided that ϕ is only assumed to be bounded. We then understand $e^h \omega^n$ as the above when ω does not have enough regularity, in particular when ω^n does not have to be strictly positive. This understanding would be important when we talk about the geodesics in the space of Kahler potentials with respect to Mabuchi's metric. Using this observation, for any given $C^{1,1}$ geodesic $\phi(t)$, the quantity

$$(2.9) \quad H(t) = \int_M \dot{\phi} e^{h(t)} \omega_t^n$$

is then well defined. When $\phi(t)$ is a smooth geodesic path of Kahler potentials, then we can compute directly that

$$(2.10) \quad \frac{d}{dt} H(t) = \int_M (\ddot{\phi} - |\nabla \dot{\phi}|^2) e^{h(t)} \omega_t^n + \int_M (|\nabla \dot{\phi}|^2 - (\dot{\phi} + a(t))^2) e^{h(t)} \omega_t^n,$$

where $a(t)$ is the time-dependent constant such that $\dot{h} = -\Delta \dot{\phi} - \dot{\phi} - a(t)$; in particular,

$$\int_M (\dot{\phi} + a(t)) e^{h(t)} \omega_t^n = 0.$$

Given the geodesic equation (with smooth assumption on Kahler potentials $\phi(t)$)

$$\ddot{\phi} - |\nabla \dot{\phi}|^2 = 0$$

and the modified Poincare inequality on Fano manifolds, we know that $H(t)$ is monotone increasing. Hence for any such a geodesic path $\phi(t)$,

$$(2.11) \quad H(0) \leq H(T), t \in [0, T]$$

Now suppose ϕ_0, ϕ_T are two smooth Kahler potentials in $[\omega_0]$ and we consider the $C^{1,1}$ geodesic segment $\phi(t)$ connecting the given two Kahler potentials. Now we claim that (2.11) still holds. We need to use the approximating geodesics established in Chen [11]. For any $\epsilon > 0$, consider the modified equation on $M \times [0, T] \times S^1$,

$$(\ddot{\phi}_\epsilon - |\nabla \dot{\phi}_\epsilon|^2) \sqrt{-1} dt \wedge ds \wedge \omega_\phi^n = \epsilon \Omega_0^{n+1}.$$

Then there exists a unique smooth path $\phi_\epsilon(t)$ of Kahler potentials such that $\phi_\epsilon(t) \rightarrow \phi(t)$ (in $C^{1,\alpha}$ for any $\alpha \in [0, 1)$) when $\epsilon \rightarrow 0$. Moreover, $\phi_\epsilon(t)$ is in $C^{1,1}$ in the sense that

$$|\phi_\epsilon|_{C^1} + \max\{|\ddot{\phi}_\epsilon|, |\nabla \dot{\phi}_\epsilon|, |\partial \bar{\partial} \phi_\epsilon|\} \leq C$$

where C is a uniform constant depending only on ϕ_0, ϕ_T and (M, ω_0) (for more details, we refer to [11]). Hence we can associate such a path

$$H_\epsilon(t) = \int_M \dot{\phi}_\epsilon e^{h_\epsilon} \omega_\epsilon^n.$$

A direct computation as above shows that

$$\frac{dH_\epsilon}{dt} = \int_M (\ddot{\phi}_\epsilon - |\nabla \dot{\phi}_\epsilon|^2) e^{h_\epsilon} \omega_\epsilon^n + \int_M (|\nabla \dot{\phi}_\epsilon|^2 - (\dot{\phi}_\epsilon + a_\epsilon(t))^2) e^{h_\epsilon} \omega_\epsilon^n > 0.$$

It then follows that

$$H_\epsilon(0) < H_\epsilon(T).$$

Certainly by approximation, we have

$$H_\epsilon(t) = H(t) + o(\epsilon).$$

The desired inequality (2.11) then follows by letting $\epsilon \rightarrow 0$. Once (2.11) is set-up, then we can use the comparison geometry in the space of Kahler potentials to prove (2.7) following the argument in ([12] Section 4.4). Since the approximating argument is similar, we keep it in brief. For any $\xi \in G$, we assume ω_0 is G-invariant such that,

$$d\theta_\xi = \iota_\xi \omega_0, \int_M e^{\theta_\xi} \omega_0^n = V.$$

Let $\omega_{\rho(t)} = \sigma_t^* \omega_0$, where $\sigma_t : M \rightarrow M$ is a one-parameter holomorphism generated by $X = -J\xi - i\xi$. Then $\rho(t), t \in (-\infty, \infty)$ is a smooth geodesic line such that

$$\dot{\rho} = -\sigma_t^* \theta_\xi.$$

The associated H function for this geodesic path is

$$H_\rho(t) = \int_M -\sigma_t^* (\theta_\xi e^{h_0} \omega_0^n) = -H(\xi).$$

For any metric $\omega = \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi_0 \in [\omega_0]$, we consider the geodesic segment $\phi(t)$ which connects ϕ_0 and ρ_T for any $T > 0$. We assume a normalization condition along the geodesic segment,

$$\int_M e^{-\phi} \omega_\phi^n = V.$$

Then following the comparison geometric argument in Chen [12], which relied on the results established in Chen [11] and Calabi-Chen [10] which asserts that the space of Kahler potentials with Mabuchi metric is actually an Alexanderov space of nonpositive curvature, we can obtain

$$(2.12) \quad \int_M |\dot{\phi}(T) - \dot{\rho}(T)|^2 \omega_{\rho(T)}^n = O(T^{-2}).$$

It then follows that $(\phi(T) = \rho(T))$,

$$|H_\rho(T) - H_\phi(T)| = \left| \int_M (\dot{\phi}(T) - \dot{\rho}(T)) e^{h_{\rho(T)}} \omega_{\rho(T)}^n \right| = O(T^{-1}).$$

By (2.11), we have

$$H_\phi(0) = \int_M \dot{\phi} e^h \omega^n \leq H_\phi(T).$$

And by the convex property (2.6), we get that

$$\mathcal{H}(\omega) \geq \int_M -\dot{\phi} e^h \omega^n = -H_\phi(0).$$

It then follows that, by letting $T \rightarrow \infty$,

$$\mathcal{H}(\omega) \geq -H_\phi(0) \geq -H_\phi(T) \rightarrow H(\xi).$$

□

As a direct consequence, we can obtain an upper bound for Perelman's μ -functional and Kahler Ricci soliton, if exists, maximizes the μ functional, without invariant assumption. Recall Perelman's W-functional is defined as

$$W(g, \tau, f) = \frac{1}{(4\pi\tau)^{-n/2}} \int_M (\tau(R + |\nabla f|^2) - n + f) e^{-f} dv_g.$$

On Fano manifolds, however, it is more convenient to let $\tau = 1/2$ and it is also convenient using (complex) geometric quantities which differ only by multiple of a constant. Hence we consider W -functional on $(M, [\omega_0])$ by

$$W(\omega, f) = \int_M (R + |\nabla f|^2 + f) e^{-f} \omega^n,$$

with the normalization condition

$$\int_M e^{-f} \omega^n = V.$$

And the μ -functional is defined to be

$$\mu(\omega) = \inf W(\omega, f).$$

Corollary 2.6. *For any $\omega \in [\omega_0]$,*

$$\mu(\omega) + \mathcal{H}(\omega) \leq nV.$$

Hence Perelman's μ -functional is bounded above by,

$$\mu(\omega) \leq nV - \sup_{\xi \in \text{Lie}(G)} H(\xi).$$

In particular, if ω_0 is a Kahler-Ricci soliton, then for any $\omega \in [\omega_0]$,

$$\mu(\omega) \leq \mu(\omega_0) = nV - \sup_{\xi \in \text{Lie}(G)} H(\xi) = nV - H(\xi_0).$$

Proof. We observe that

$$\mu(\omega) \leq W(\omega, -h) = \int_M ((n + \Delta h) + |\nabla h|^2 - h) e^h \omega^n = nV - \mathcal{H}(\omega).$$

When ω_0 is a Kahler-Ricci soliton, then h_0 is the Hamiltonian of ξ_0 (the imaginary part of the extremal vector field),

$$\mathcal{H}(\omega_0) = \int_M h_0 e^{h_0} \omega_0^n = H(\xi_0) = \sup_{\xi \in \text{Lie}(G)} H(\xi)$$

and it is well known that

$$\mu(\omega_0) = W(\omega, -h_0) = nV - H(\xi_0).$$

□

For invariant metrics, such an upper bound was proved in a recent paper by Tian-Zhang-Zhang-Zhu [32] and the authors proved that the Kahler-Ricci flow with any invariant initial metric actually maximizes μ functional asymptotically provided that the modified Mabuchi functional is bounded below, and it is used to give an alternative proof of convergence of the Kahler-Ricci flow for invariant metric assuming the existence of Kahler-Ricci soliton. It is a belief that if Kahler-Ricci soliton exists, then the Kahler-Ricci flow would actually converge to the soliton, without invariant assumption on initial metric. In [32] the authors proposed a conjecture (Conjecture 3.3), which suggests how one might prove such a convergent result. Corollary 2.6 seems to strengthen this belief.

The proof of Theorem 2.3 relied on the monotonicity of $H(t)$ (2.11) along geodesic segments and the comparison geometric argument for geodesic segments used in [12]. The monotonicity of $H(t)$ is actually a special case of Berndtsson [5, 6] (for smooth (or $C^{1,1}$) geodesic, this convexity is actually a direct consequence of the well-known modified Poincare inequality on Fano manifolds). We state his results for convenience (for more details including notations, we refer to [6]).

Let X be a projective manifold with semi-negative canonical line bundle ($-K_X \geq 0$) of dimension n and let U be a domain in \mathbb{C} . We use w to denote the coordinate in U .

Theorem 2.7 (Berndtsson). *Assume that $-K_X \geq 0$ and let ϕ_w be a curve of metrics on $-K_X$ such that*

$$\sqrt{-1}\partial\bar{\partial}_{w,X}\phi_w \geq 0$$

in the sense of current on $X \times U$, Then

$$F(w) := -\log \int_X e^{-\phi_w}$$

is subharmonic in U . If ϕ_w depends only on t , the real part of w ($w = t + is$), then F is convex in t . Moreover, assume $H^{0,1}(X) = 0$ and ϕ_t is uniformly bounded in the sense that there is a smooth metric ψ on $-K_X$ such that $|\psi - \phi_t| \leq C$. Then if $F(t)$ is a linear function of t in the neighborhood of $0 \in U$, then there exists a holomorphic vector field (possibly t -dependent) V on X with flow σ_t such that

$$\sigma_t^*(\partial\bar{\partial}\phi_t) = \partial\bar{\partial}\phi_0.$$

3. KAHLER-EINSTEIN METRIC AND GEODESIC STABILITY

Donaldson [16] formulated a conjecture relating the existence of constant scalar curvature metric in $(M, [\omega_0])$ to the geometry of the space of Kahler potentials, as an analogue of the Hilbert-Mumford criterion for stability of finite dimensional Kahler quotient theory.

Conjecture 3.1 (Donaldson). *The following are equivalent:*

- (1) *There is no constant scalar metric in $(M, [\omega_0])$.*
- (2) *There is a geodesic ray $\phi(t), t \in [0, 1)$ such that the derivative of Mabuchi's K -energy*

$$\frac{d\mathcal{K}}{dt} = - \int_M \dot{\phi}(R - \underline{R})\omega_\phi^n < 0$$

for all $t \in [0, \infty)$.

- (3) *For any Kahler potential ϕ , there exists a geodesic ray as in (2) starting at ϕ .*

The conjecture remains open in both directions (see [13] for partial results). One technical difficulty is that the geodesic rays do not have enough regularity to talk about the derivative of \mathcal{K} -energy in general. But \mathcal{F} -functional and its derivative make sense even for geodesics with only bounded potential. Hence when $(M, [\omega_0])$ is Fano, in view of the relation of \mathcal{F} -functional and Futaki invariant, and the relation of its convexity and existence of Kahler-Einstein metric, it is very natural to ask,

Conjecture 3.2. *Let $(M, [\omega_0])$ be a Fano manifold. The following are equivalent,*

- (1) *There is no Kahler-Einstein metric in $(M, [\omega_0])$.*
- (2) *There is a geodesic ray $\phi(t), t \in [0, \infty)$ such that the derivative of \mathcal{F} -functional is strictly negative for all $t \in [0, \infty)$.*
- (3) *For any Kahler potential ϕ , there exists a geodesic ray as in (2) starting at ϕ .*

We can also formulate the conjecture as follows,

Conjecture 3.3. *Let $(M, [\omega_0])$ be a Fano manifold. Then the following are equivalent*

- (1) *There exists a Kahler-Einstein metric in $(M, [\omega_0])$.*
- (2) *There exists a Kahler potential ϕ , for every geodesic ray $\phi(t), t \in [0, \infty)$ starting at ϕ satisfies that for some $t_0 \in (0, \infty)$,*

$$\frac{d\mathcal{F}}{dt}(t_0) \geq 0.$$

- (3) *For every point $\phi \in \mathcal{H}$ and for every geodesic ray starting at ϕ it satisfies (2).*

The above conjecture is clearly a direct modification of Donaldson's conjecture. The advantage of using \mathcal{F} -functional is mainly for technical reason as mentioned above. The regularity of geodesic rays is not specified yet in the above conjectures. It is actually an interesting technical problem to be precise on the regularity.

Recall the space of Kahler potentials in $(M, [\omega_0])$ is defined by

$$\mathcal{P} = \{\phi \in C^\infty : \omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0\}$$

Clearly adding a constant to a given Kahler potential does not change the corresponding Kahler metric and we define \mathcal{P}_0 to be the space of Kahler metrics in $(M, [\omega_0])$ by modulo addition of constants,

$$\mathcal{P}_0 = \{\omega \in [\omega_0] : \omega > 0\}.$$

We need to consider generalized Kahler potentials. The minimum requirement is that $\omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi$ defines a closed positive $(1,1)$ current. If $\phi \in L^\infty$, then ω_ϕ^n is a well-defined volume form such that

$$\int_M \omega_\phi^n = \int_M \omega_0^n = V.$$

In this case we call ϕ a bounded Kahler potential and we denote the set of all bounded Kahler potentials as \mathcal{P}_∞ .

The Mabuchi metric [23] on \mathcal{P} is defined as, for $\psi_1, \psi_2 \in T_\phi \mathcal{P}$,

$$\langle \psi_1, \psi_2 \rangle_\phi = \int_M \psi_1 \psi_2 \omega_\phi^n.$$

For any path $\phi(t) \in \mathcal{P}$, then the geodesic equation is given by

$$\ddot{\phi} - |\nabla \dot{\phi}|_{\omega_\phi}^2 = 0.$$

For any interval I in \mathbb{R} , denote $U = I \times S^1$. We use (z, w) to denote points on $M \times U$. Then the geodesic equation is equivalent to the homogeneous complex Monge-Ampere equation (assuming for each w , ϕ defines a strictly positive Kahler metric),

$$(3.1) \quad \Omega_\phi^{n+1} = 0,$$

where $\Omega_\phi = \pi^* \omega_0 + \partial\bar{\partial}_{w,z}\phi$, $\pi : M \times U \rightarrow M$ is the projection onto M and ϕ is regarded as a S^1 invariant function on $M \times U$. A fundamental result of Chen [11] asserts that for $I = [0, 1]$ and $\phi_0, \phi_1 \in \mathcal{P}$, there exists a unique $C^{1,1}$ solution of (3.1) in the sense that

$$(3.2) \quad \|\phi\|_{1,1} := \|\phi\|_{C^1} + \max\{|\partial\bar{\partial}_{w,z}\phi|\} \leq C.$$

Here $\phi(w, z)$ is regarded as a function on $M \times U$. Note that Chen's estimates rely on the fact that the two end points are actually in \mathcal{P} .

Sometimes it is useful to consider generalized $C^{1,1}$ Kahler potentials defined as

$$\mathcal{P}_{1,1} = \{\phi : \omega_\phi \geq 0, \|\phi\|_{C^1} < \infty, 0 \leq n + \Delta\phi < \infty\}.$$

We also define the (weak) $C^{1,1}$ norm on M as follows (ϕ is a function on M), fixing a background metric,

$$\|\phi\|_{1,1}^w = \|\phi\|_{C^1} + \max |\Delta\phi|$$

If $I = [0, \infty)$ and $\phi(t)$ satisfies (3.1), then $\phi(t)$ is called a geodesic ray. It is called a bounded geodesic ray if $\phi(t) \in \mathcal{P}_\infty$ for each t and it is called a $C^{1,1}$ geodesic ray if $\phi(t) \in \mathcal{P}_{1,1}$ for each t . Note that we do not specify any condition on ϕ_{tt} (or $\phi_{w\bar{w}}$). We define geodesic stability as follows (we refer to Chen [13] for the geodesic stability in terms of \mathcal{K} -energy in general case),

Definition 3.4. Let $(M, [\omega_0])$ be a Fano manifold. Then we call it is geodesic semistable if there exists a point ϕ in \mathcal{P} , for any $C^{1,1}$ geodesic ray $\phi(t)$ starting at ϕ ,

$$\lim_{t \rightarrow \infty} \frac{d\mathcal{F}}{dt} \geq 0.$$

We call it is called strong geodesic semistable if for every point ϕ in \mathcal{P} , for any $C^{1,1}$ geodesic ray $\phi(t)$ starting at ϕ ,

$$\lim_{t \rightarrow \infty} \frac{d\mathcal{F}}{dt} \geq 0.$$

It is called geodesic stable if there is a point $\phi \in \mathcal{P}$, for any $C^{1,1}$ geodesic ray $\phi(t)$ starting at ϕ , there exists t_0 (t_0 can clearly be dependent of the geodesic) such that

$$\frac{d\mathcal{F}}{dt}(t_0) \geq 0.$$

Similarly we can define strong geodesic stability.

We can also define the geodesic stability in terms of bounded geodesic rays.

The following is straightforward,

Proposition 3.5. *If $(M, [\omega_0])$ admits a Kahler-Einstein metric, then $(M, [\omega_0])$ is geodesic stable for bounded geodesic rays.*

Proof. Choose ω_0 as the base point, then \mathcal{F} functional obtains its minimum at ω_0 and the statement follows from the convexity of \mathcal{F} -functional. \square

Clearly strong geodesic stability implies geodesic stability and it might be an interesting problem to prove the equivalence. It would be more interesting to prove that the existence of Kahler-Einstein metric implies strong geodesic stability. By assuming a technical condition, we can prove that the existence of Kahler-Einstein metric implies strong geodesic semistability. From the proof we believe two versions of semistability are equivalent.

In the following, we need to assume that the $C^{1,1}$ geodesic rays considered have the following property, there exists a sequence of time $t_k \rightarrow \infty$ such that $\phi(t_k) \in \mathcal{P}$.

Theorem 3.6. *If $(M, [\omega_0])$ admits a Kahler-Einstein metric, then $(M, [\omega_0])$ is strong geodesic stable for $C^{1,1}$ geodesic rays which satisfy the above assumption.*

Proof. First we need that for any $C^{1,1}$ (more generally for bounded) geodesic ray $\phi(t)$, \mathcal{F} -functional is well-defined and it is convex along the ray (see Berndtsson [6]). Recall \mathcal{F} -functional can be defined as

$$\mathcal{F}_{\omega_0}(\phi) = \mathcal{E}_0(\phi) - \log \left(\frac{1}{V} \int_M e^{h_0 - \phi} \omega_0^n \right),$$

where \mathcal{E}_0 is the well-known energy functional

$$\mathcal{E}_0(\phi) = -\frac{1}{(n+1)V} \sum_{j=1}^n \int_M \phi \omega_0^j \wedge \omega_\phi^{n-j}$$

and it can be characterized by its derivative along any (C^1 for example) path $\phi(t)$

$$\frac{d\mathcal{E}_0(\phi(t))}{dt} = - \int_M \dot{\phi} \omega_\phi^n.$$

Note \mathcal{F} -functional satisfies cocycle condition and we simply write $\mathcal{F}_{\omega_0} = \mathcal{F}$. It was shown actually $\mathcal{E}_0(\phi)$ is linear along bounded geodesic segments [6] and

$$-\log \left(\frac{1}{V} \int_M e^{h_0 - \phi} \omega_0^n \right).$$

is convex along bounded geodesic segments. Hence \mathcal{F} -functional is convex along bounded geodesics.

Suppose ω_0 is a Kahler-Einstein metric. Let $\phi(t)$ be a $C^{1,1}$ geodesic ray with initial point ϕ_0 . Now we consider the geodesic segment $\rho(t), t \in [0, T]$ from 0 (the metric ω_0) to $\phi(T)$. Now we need to use a comparison geometric argument as in Chen [12]. Following Calabi-Chen [10], \mathcal{P} is a metric space with nonpositive curvature in the sense of Alexanderov with Mabuchi's metric. However, it remains unclear that $\mathcal{P}_{1,1}$ is even a metric space.

By our assumption, we can choose a sequence of time $T_k \rightarrow \infty$, such that $\phi(T_k) \in \mathcal{P}$. In the following we take $T = T_k$ for k large enough. Consider the triangle in \mathcal{P} with vertices $A = 0, B = \phi_0, C = \phi(T)$. Denote the distance $|AB| = d$ and along geodesic segment $\phi(t)$, $\int_M \dot{\phi}^2 \omega_\phi$ is constant and we assume both are nonzero for AC, BC . Then we have

$$|AC|^2 = T^2 \int_M \dot{\rho}^2 \omega_\rho^n, |BC|^2 = T^2 \int_M \dot{\phi}^2 \omega_\phi^n.$$

Let $\tilde{A}, \tilde{B}, \tilde{C}$ be the vertices of an Euclidean triangle with the same length as ABC and denote the angle at \tilde{C} by $\tilde{\theta}$. Then

$$\cos \tilde{\theta} = (|AC|^2 + |BC|^2 - d^2) / 2|AC||BC|.$$

The tangent vector at C for AC is given by $\dot{\rho}(T)$, for BC by $\dot{\phi}(T)$. The inner product of two vectors is then

$$(\dot{\phi}(T), \dot{\rho}(T)) = \int_M \dot{\phi} \dot{\rho} \omega_{\phi(T)}^n.$$

Hence the angle θ formed at C of the triangle ABC is given by

$$\cos \theta = \frac{\int_M \dot{\phi} \dot{\rho} \omega_{\phi(T)}^n}{\left(\int_M \dot{\phi}^2 \omega_{\phi(T)}^n \int_M \dot{\rho}^2 \omega_{\phi(T)}^n \right)^{1/2}}$$

Since \mathcal{P} has nonpositive curvature, we know $\theta \leq \tilde{\theta}$, hence $0 < \cos \tilde{\theta} \leq \cos \theta \leq 1$ (note that we consider T really large, hence $\tilde{\theta}$ is small). It follows that

$$(3.3) \quad T^2 \int_M (\dot{\phi}(T) - \dot{\rho}(T))^2 \omega_{\phi(T)}^n \leq d^2.$$

Now we compute

$$\begin{aligned} \frac{d\mathcal{F}(\phi(t))}{dt}(T) &= - \int_M \dot{\phi}(T) \omega_{\phi(T)}^n + \frac{V \int_M \dot{\phi}(T) e^{h_0 - \phi(T)} \omega_0^n}{\int_M e^{h_0 - \phi(T)} \omega_0^n} \\ \frac{d\mathcal{F}(\rho(t))}{dt}(T) &= - \int_M \dot{\rho}(T) \omega_{\phi(T)}^n + \frac{V \int_M \dot{\rho}(T) e^{h_0 - \phi(T)} \omega_0^n}{\int_M e^{h_0 - \phi(T)} \omega_0^n} \end{aligned}$$

To proceed we need to consider normalization of Kahler potentials. Note that \mathcal{F} -functional does not depend on the normalization (addition of constant function on M). For any geodesic ray (segment) $\phi(t)$, adding a linear function of t ($\phi(t) + at + b$, a, b are constants) still gives geodesic ray (segment). In particular, \mathcal{F} -functional and its derivative (with respect to t) would remain the same. Using this observation, we can then choose a normalization condition (depending on T) for $\phi(t)$ and $\rho(t)$ such that

$$\begin{aligned}\int_M \dot{\phi}(T) e^{h_0 - \phi(T)} \omega_0^n &= 0 \\ \int_M \dot{\rho}(T) e^{h_0 - \phi(T)} \omega_0^n &= 0\end{aligned}$$

Clearly (3.3) remains valid for any such normalization and it then follows that

$$\left| \frac{d\mathcal{F}(\rho(t))}{dt}(T) - \frac{d\mathcal{F}(\phi(t))}{dt}(T) \right| = O(T^{-1}).$$

Then note that \mathcal{F} functional takes its minimum at Kahler-Einstein metrics and hence

$$\frac{d\mathcal{F}(\rho(t))}{dt}(T) \geq 0.$$

Let $T \rightarrow \infty$, we then get

$$\lim_{t \rightarrow \infty} \frac{d\mathcal{F}(\phi(t))}{dt} \geq 0.$$

□

It would be interesting to prove that Theorem 3.6 actually holds without the extra assumption, and also even for only bounded geodesic rays. Note that given two end points $\phi_0, \phi_1 \in \mathcal{P}_\infty$, there is a bounded geodesic segment $\phi(t)$ (see Berndtsson [6] for example) and $\dot{\phi}(t)$ is uniformly bounded depending only on the L^∞ of two end points. In particular the length of geodesic segment is still well defined for bounded geodesic segments. However, it is not clearly that $\mathcal{P}_{1,1}$ (or \mathcal{P}_∞) is a metric space. The point is that Chen's estimates (3.2) relies on the fact that two end points are in \mathcal{P} —the nondegeneracy and sufficient regularity of two boundary points are important for the uniform estimate ϕ_{tt} , in particular; it is also important to note that the triangle inequality (or the geodesic minimizing along all curves) also relies on the nondegeneracy of two boundary points. Hence we would like to ask

Question 3.7. *Is $\mathcal{P}_{1,1}$ (or \mathcal{P}_∞) a metric space? What is the relation with the metric completion of \mathcal{P} ?*

More specifically,

Question 3.8. *For any $\phi \in \mathcal{P}_{1,1}$, note that one can choose a sequence $\phi_k \in \mathcal{P}$ such that $\phi_k \rightarrow \phi$ in $C^{1,\alpha}$ and such that $\Delta\phi_k$ bounded. Then is it true $d(\phi, \phi_k) \rightarrow 0$ when $k \rightarrow \infty$? In particular, for $\phi, \psi \in \mathcal{P}_{1,1}$, we choose two approximating sequences ϕ_k, ψ_k as above, is it true*

$$d(\phi_k, \psi_k) \rightarrow d(\phi, \psi)?$$

Clearly if the answer is affirmative, then $\mathcal{P}_{1,1}$ is a metric space with non-positive curvature in the sense of Alexanderov by an approximation argument. And then we do not need extra assumption in Theorem 3.6 and moreover, two versions of geodesic semistability would be equivalent by similar arguments.

We can also ask similar questions for bounded Kahler potentials,

Question 3.9. *For any $\phi \in \mathcal{P}_\infty$, note that one can choose a decreasing sequence $\phi_k \in \mathcal{P}$ such that $\phi_k \rightarrow \phi$ (see [7] for example). Then is it true $d(\phi, \phi_k) \rightarrow 0$ when $k \rightarrow \infty$?*

Remark 3.10. Upon the preparation of the present paper, Berman [4] proved that the existence of Kahler-Einstein metric implies K-polystability (for Q -Fano variety) using the convexity of \mathcal{F} -functional. One of the key points in his proof is to compute Donaldson-Futaki invariant in terms of asymptotes of \mathcal{F} -functional along the geodesic rays associated with a given test configuration, which shows that K-stability is essentially equivalent to geodesic stability. We hope our discussion regarding geodesic stability is still of its own interest.

Remark 3.11. Two different versions of geodesic stability are conjectured to be equivalent (see Conjecture 3.3). However it does not seem to be straightforward to prove this even the above discussion regarding $\mathcal{P}_{1,1}$ is affirmative. Hopefully the strong geodesic stability would be helpful to prove existence problem of Kahler-Einstein metrics.

4. THE MODIFIED \mathcal{F} -FUNCTIONAL AND KAHLER-RICCI SOLITON

The discussion for Kahler-Einstein metrics in last section can be extended to soliton case without essential modification. We can then formulate a version of geodesic stability for Kahler-Ricci soliton and prove that the existence implies geodesic semistability.

Recall the modified \mathcal{F} functional is defined by Tian-Zhu [30] as follows, given a holomorphic vector field X ,

$$(4.1) \quad \mathcal{F}_X(\phi) = -\frac{1}{V} \int_0^1 \int_M \dot{\phi}_t e^{\theta_X(\phi_t)} \omega_{\phi_t}^n dt - \log \left(\frac{1}{V} \int_M e^{h_g - \phi} \omega^n \right),$$

where h is the Ricci potential of ω , and $\theta_X(\omega)$ is the potential of X with respect to ω ($\iota_X \omega = \sqrt{-1} \bar{\partial} \theta_X$), both satisfying the normalization,

$$\int_M e^h \omega^n = \int_M e^{\theta_X} \omega^n = \int_M \omega^n = V.$$

The first variation formula of \mathcal{F}_X is given by,

$$(4.2) \quad \frac{d\mathcal{F}_X}{dt} = -\frac{1}{V} \int_M \dot{\phi} e^{\theta_X(\phi)} \omega_\phi^n + \left(\int_M e^{h-\phi} \omega^n \right)^{-1} \int_M \dot{\phi} e^{h-\phi} \omega^n.$$

The Euler-Lagragian equation is given by

$$e^{\theta_X(\phi)} \omega_\phi^n = V e^{h-\phi} \omega^n \left(\int_M e^{h-\phi} \omega^n \right)^{-1}.$$

This is exactly the equation for Kähler-Ricci soliton with respect to the vector field X . By choosing a normalization condition of ϕ , we can assume

$$(4.3) \quad \int_M e^{h-\phi} \omega^n = V.$$

We can compute the second derivative of \mathcal{F} and prove its convexity along smooth geodesics of Kähler potentials,

Proposition 4.1. *The second variation of \mathcal{F} is given by*

$$(4.4) \quad \frac{d^2 \mathcal{F}_X}{dt^2} = -\frac{1}{V} \int_M (\ddot{\phi} - |\nabla \dot{\phi}|_\phi^2) (e^{\theta_X(\phi)} \omega_\phi^n - e^{h-\phi} \omega^n) + A,$$

where

$$A = \frac{1}{V} \int_M (|\nabla \dot{\phi}|_\phi^2 - |\dot{\phi}|^2) e^{h_\phi} \omega_\phi^n.$$

Proof. Note that we assume the normalization condition (4.3). Hence for the Ricci potential h_ϕ , we have

$$(4.5) \quad h_\phi + \log \frac{\omega_\phi^n}{\omega^n} = h - \phi.$$

We also have the normalization condition for $\dot{\phi}$,

$$(4.6) \quad \int_M \dot{\phi} e^{h_\phi} \omega_\phi^n = 0.$$

Using (4.3) and (4.5), (4.4) follows from a direct computation by taking derivative of (4.2). Note that given (4.6), by the modified Poincaré inequality on Fano manifolds,

$$A = \frac{1}{V} \int_M (|\nabla \dot{\phi}|_\phi^2 - |\dot{\phi}|^2) e^{h_\phi} \omega_\phi^n \geq 0.$$

The equality holds if and only if $\nabla \dot{\phi}$ gives a holomorphic vector field. \square

Invoking Berndtsson [6], we have the following,

Proposition 4.2. *The modified \mathcal{F}_X functional is convex along any $C^{1,1}$ geodesic $\phi(t) \in \mathcal{P}_{1,1}$. If $\mathcal{F}_X(\phi(t))$ is a linear function in t , then there exists a one-parameter holomorphism $\sigma_t : M \rightarrow M$ such that*

$$\sigma_{\phi(t)} = \sigma_t^* \omega_{\phi_0}.$$

Proof. The only thing we need in addition is that $\theta_X(\phi)$ is uniformly bounded. This follows from [34] and the proof can be extended directly to $C^{1,1}$ potentials. The statement then follows Berndtsson [6]. \square

Remark 4.3. As a consequence of this convexity one can give a proof of Tian-Zhu's result [30] on the uniqueness of Kähler-Ricci soliton in $(M, [\omega_0])$. The only additional fact we need is that the extremal vector field for Kähler-Ricci soliton is unique up to automorphisms [31]. In Kähler-Einstein case, Berman [3] and Berndtsson [6] have already given a new proof of Bando-Mabuchi's uniqueness theorem [2] using such convexity directly. We learned from Song Sun that Berndtsson can extend his results in [6] to give a new proof of Tian-Zhu's uniqueness theorem.

We can define an invariant using the derivative of \mathcal{F}_X functional. For any $\xi \in \text{Lie}(G)$, we define

$$(4.7) \quad F_X(\xi) = \int_M \theta_\xi (e^h - e^{\theta x}) \omega^n.$$

When there exists a Kahler-Ricci soliton with X as extremal vector field, then $F_X(\xi) = 0$ for any $\xi \in \text{Lie}(G)$. Actually it is just the modified Futaki invariant defined by Tian-Zhu [31] and it is straightforward to see if we choose a normalization

$$\int_M \tilde{\theta}_\xi e^h = 0.$$

Note that the modified Futaki invariant is related to the Futaki invariant as follows,

$$F_X(\xi) = F(\xi) - \int_M \theta_\xi (e^{\theta x} - 1) \omega^n.$$

We can then formulate a version of geodesic stability regarding the existence of Kahler-Ricci soliton in terms of modified \mathcal{F}_X -functional.

Conjecture 4.4. *Let $(M, [\omega_0])$ be a Fano manifold. The following are equivalent,*

- (1) *There is a Kahler-Ricci soliton on M with extremal vector field X .*
- (2) *There is a point $\phi \in \mathcal{P}$, such that for any geodesic ray $\phi(t)$ starting at ϕ , such that the derivative of \mathcal{F}_X -functional is nonnegative for some $t_0 \in [0, \infty)$.*
- (3) *For any geodesic ray $\phi(t), t \in [0, \infty)$ such that the derivative of \mathcal{F}_X -functional is nonnegative for some $t_0 \in [0, \infty)$.*

Clearly without any essential modification (see Theorem 3.6), we can obtain

Proposition 4.5. *If $(M, [\omega_0])$ admits a Kahler-Ricci soliton, then $(M, [\omega_0])$ is geodesic stable. With the same assumption as in Theorem 3.6, then $(M, [\omega_0])$ is strong geodesic semistable.*

5. DISCUSSIONS

There are several further questions that seem to be interesting.

- (1) Geodesic stability certainly implies \mathcal{F} -functional is bounded (proper) along any geodesic ray (geodesic line). Can one show that geodesic stability implies \mathcal{F} -functional bounded below? It seems that the discussion in [13] in terms of \mathcal{K} -energy would be very helpful.
- (2) Give an algebro-geometric description of stability for Kahler-Ricci soliton.
- (3) Study $\mathcal{P}_{1,1}$ (or \mathcal{P}_∞) and its relation with the metric completion $\bar{\mathcal{P}}$ (see Question 3.8, 3.9 for example).

Remark 5.1. Motivated in part by the questions above, we can prove the regularity of geodesic segment with two end points $\phi_0, \phi_1 \in \mathcal{P}_{1,1}$: for each t , $\phi(t) \in \mathcal{P}_{1,1}$. The details will appear elsewhere.

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